

INSTANTONS AND THE LARGE N_C LIMIT, A.D. 2004

T. SCHÄFER

*Department of Physics
North Carolina State University
Raleigh, NC 27695*

In this contribution we discuss our current understanding of instanton effects in the large N_c limit of QCD. We argue that the instanton liquid can have a smooth large N_c limit which is in agreement with scaling relations derived from Feynman diagrams. We also discuss certain limits of QCD, like the case of high baryon density, in which the Witten-Veneziano relation can be derived from QCD and is saturated by instantons.

1. Introduction

In the limit in which the masses of the light up, down and strange quarks are taken to zero, and the masses of the heavy quarks are taken to infinity QCD is a parameter free theory. This is one of the aspects of QCD that make it such a beautiful theory, as it implies that all dimensionful numbers, like the masses and radii of hadrons, can be expressed in terms of a single dimensionful quantity, Λ_{QCD} . It also implies, however, that there is no expansion parameter that can be used to perform systematic calculations.

Many years ago 't Hooft suggested to consider the limit in which the number of colors, N_c , is large and to use $1/N_c$ as an expansion parameter¹. In order to keep the QCD scale parameter fixed we have to take the $N_c \rightarrow \infty$ limit with the 't Hooft parameter $\lambda = g^2 N_c$ constant. In the large N_c limit the perturbative expansion in Feynman diagrams is replaced by an expansion in the genus of the two-dimensional Riemann surface spanned by the diagrams. This result fits very well with the idea that the large N_c limit of Yang-Mills theory is equivalent to a string theory. For $N = 4$ SUSY Yang-Mills theory an explicit realization of this idea is provided by the AdS/CFT correspondence, but in the case of QCD the precise form of the string theory is not known.

An interesting problem arises if we consider the fate of the axial anomaly

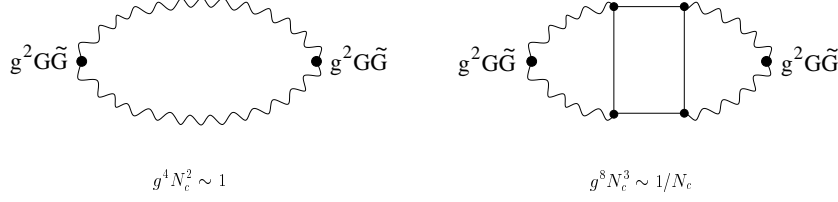


Figure 1. The diagrams on the left and right show the large N_c scaling of typical diagrams that contribute to the topological charge correlator in pure gauge QCD and in QCD with light fermions.

in the large N_c limit. For this purpose we add a θ term

$$L = \frac{ig^2\theta}{32\pi^2} G_{\mu\nu}^a \tilde{G}_{\mu\nu}^a, \quad (1)$$

to the QCD lagrangian. The θ term is a total derivative and in perturbation theory physics is independent of θ . Witten suggested that non-perturbative effects generate θ -dependence in the pure gauge theory and that the topological susceptibility,

$$\chi_{top} = \left. \frac{d^2 E}{d\theta^2} \right|_{\theta=0}, \quad (2)$$

is $O(1)$ in the large N_c limit^{2,3}. This suggestion was originally based on the fact that perturbative contributions to the topological charge correlator scale as N_c^0 in the large N_c limit, see Fig. 1a. Recently, Witten provided additional evidence for this conjecture using the AdS/CFT correspondence³. The scaling behavior $\chi_{top} \sim N_c^0$ was also observed in lattice simulations of pure gauge QCD for⁴ $N_c = 2, \dots, 6$.

Naive N_c counting implies that the contribution of fermions to χ_{top} is subleading in $1/N_c$. We know, however, that there is no θ -dependence in QCD with massless fermions. This implies that the topological susceptibility receives a contribution related to fermions that cancels the pure gauge result, see Fig. 1b. Witten argued that this apparent contradiction can be resolved if the mass of the η' meson scales as $N_c^{-1/2}$ in the large N_c limit. Witten and Veneziano derived a relation between the mass of the η' and the topological susceptibility in pure gauge theory^{5,6,7,8}

$$f_\pi^2 m_{\eta'}^2 = 2N_F \chi_{top}|_{no \text{ quarks}}. \quad (3)$$

Using $\chi_{top} = O(1)$ and $f_\pi^2 = O(N_c)$ we observe that indeed $m_{\eta'}^2 = O(1/N_c)$.

The θ -dependence of vacuum energy is related to topological properties of QCD. In the semi-classical approximation these features can be described in terms of instantons. Instantons are localized field configurations that carry topological charge⁹

$$Q_{top} = \frac{g^2}{32\pi^2} \int d^4x G_{\mu\nu}^a \tilde{G}_{\mu\nu}^a = \pm 1. \quad (4)$$

If the coupling is small then the density of instantons scales as $\exp(-8\pi^2/g^2)$. In this limit instantons form a dilute, weakly interacting gas. The topological susceptibility is

$$\chi_{top} = \lim_{V \rightarrow \infty} \frac{\langle Q_{top}^2 \rangle}{V} \simeq \frac{N}{V}, \quad (5)$$

where $N = N_+ + N_-$ is the number of instantons and anti-instantons and V is the volume. In 1978 Witten pointed out that this result implies that the contribution of instantons to the topological susceptibility scales as $\exp(-1/g^2) \sim \exp(-N_c)$ which seems to contradict the assumption² $\chi_{top} = O(1)$.

This argument is a little oversimplified, since instantons in QCD come in all sizes, and only small instantons are exponentially suppressed. We will come back to this problem in Sect. 4. Before we do so, we would like to comment on phenomenological consequences of equ. (5). Using the experimental values of f_π and $m_{\eta'}$ the Witten-Veneziano relation implies $\chi_{top} \simeq (200 \text{ MeV})^4$ for $N_c = 3$. If the topological susceptibility is saturated by a dilute gas of instantons, this value corresponds to a density $(N/V) \simeq 1 \text{ fm}^{-4}$. An estimate of the typical instanton size can be obtained by using the perturbative instanton size distribution and integrating it up to the phenomenological value of (N/V) . This leads to a value of $\rho \simeq 1/3 \text{ fm}$. These two numbers form the basis of a successful picture of the QCD vacuum, usually called the instanton liquid model^{10,11,12}. The instanton model not only accounts for topological properties of the QCD vacuum, but also describes chiral symmetry breaking and the correlation functions of light hadrons^{13,14,15}.

Topological properties of the QCD vacuum have also been studied in lattice QCD. It was found that the topological susceptibility in pure gauge QCD is¹⁶ $\chi_{top} \simeq (200 \text{ MeV})^4$, as predicted by the Witten-Veneziano relation. It was also observed that the topological susceptibility is stable under cooling, and appears to be dominated by semi-classical configurations. Lattice simulations also seem to confirm the values of the key parameters of the instanton liquid¹⁷, $(N/V) \simeq 1 \text{ fm}^{-4}$ and $\rho \simeq 1/3 \text{ fm}$.

2. Instantons and the Witten-Veneziano relation

Before we discuss the N_c scaling behavior we would like to study the mechanism for topological charge screening and the mass of the η' in the instanton model. We will assume that instantons are small, $\rho\Lambda_{QCD} \ll 1$, and that the instanton liquid is dilute, $\rho^4 N/V \ll 1$. As we shall see below, these assumptions can be rigorously justified in the case of QCD at large baryon density. At zero density, however, this is a model assumption.

The partition function of the instanton ensemble can be written as^{18,19,20}

$$Z = \sum_{N_+, N_-} \frac{\mu_0^{N_+ + N_-}}{N_+! N_-!} \prod_i^{N_+ + N_-} d^4 z_i \exp(-S_{eff}). \quad (6)$$

where μ_0 is the partition function of a single instanton and the effective action is given by

$$S_{eff} = i \int d^4 x \frac{\sqrt{2N_f}}{f_\pi} \eta_0 Q + \int d^4 x L(\eta_0, \eta_8). \quad (7)$$

Here, the topological charge density is $Q(x) = \sum Q_i \delta(x - z_i)$ and $L(\eta_0, \eta_8)$ is the flavor singlet sector of the pseudoscalar meson lagrangian

$$L = \frac{1}{2} ((\partial_\mu \eta)^2 + (\partial_\mu \eta_8)^2) + \frac{1}{2} \left(\frac{4}{3} m_K^2 - \frac{1}{3} m_\pi^2 \right) \eta_8^2 \\ + \frac{1}{2} \left(\frac{2}{3} m_K^2 + \frac{1}{3} m_\pi^2 \right) \eta_0^2 + \frac{2\sqrt{2}}{3} (m_\pi^2 - m_K^2) \eta_0 \eta_8. \quad (8)$$

The meson lagrangian arises from bosonizing the fermionic interaction between instantons. The pion decay constant f_π is determined by the solution of a saddle point equation, see Sect. IV.G. in the review article¹⁴ for more details. The meson masses m_π and m_K satisfy Gell-Mann-Oakes-Renner relations.

The partition function equ. (6) describes a system of charges interacting through the exchange of almost massless eta mesons. The physics of this system is very similar to that of a Coulomb gas. We expect, in particular, that topological charge gets screened and that the eta meson acquires a mass. We can show this explicitly by performing the sum in equ. (6) and expanding the resulting cosine function to second order in the fields. The topological charge correlator is given by

$$\langle Q(x) Q(0) \rangle = \left(\frac{N}{V} \right) \left\{ \delta^4(x) - \frac{2N_f N}{f_\pi^2 V} [\cos^2(\phi) D(m_{\eta'}, x) \right. \\ \left. + \sin^2(\phi) D(m_\eta, x)] \right\}, \quad (9)$$

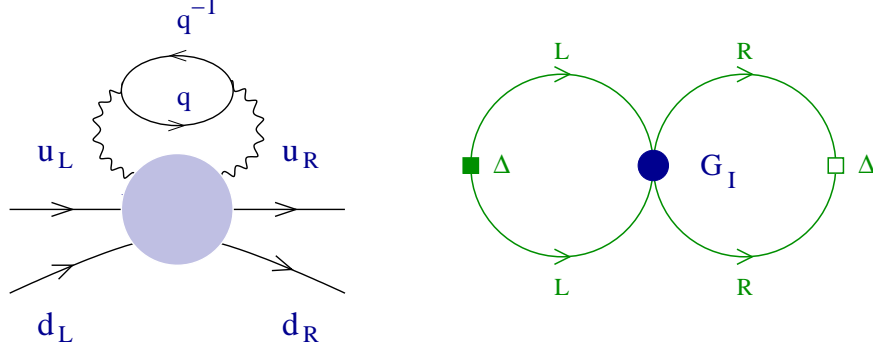


Figure 2. The figure on the left shows the instanton induced quark interaction in $N_F = 2$ QCD at non-zero baryon density. Scattering on particle-hole qq^{-1} pairs leads to gauge field screening and suppresses large instantons. The figure on the right shows the instanton contribution to the vacuum energy at large baryon density. The squares denote insertions of the diquark condensates $\langle\psi\psi\rangle$ and $\langle\bar{\psi}\bar{\psi}\rangle$.

where $D(m, x) = mK_1(mx)/(4\pi^2 x)$ is the euclidean space propagator of a scalar particle. The delta-function is the contribution from a single instanton, while the other terms are the contribution of the screening cloud. The η and η' mass satisfy the Witten-Veneziano relation

$$f_\pi^2 (m_{\eta'}^2 + m_\eta^2 - 2m_K^2) = 2N_F \left(\frac{N}{V} \right), \quad (10)$$

and ϕ is the $\eta - \eta'$ mixing angle. The coefficient of the delta-function is the topological susceptibility in the pure gauge theory. The topological susceptibility in the full theory can be calculated using equ. (9). The result is

$$\chi_{top} = \frac{f_\pi^2}{2N_f} m_{top}^2 \left\{ 1 - \frac{(\frac{4}{3}m_K^2 - \frac{1}{3}m_\pi^2)m_{top}^2}{(\frac{4}{3}m_K^2 - \frac{1}{3}m_\pi^2)m_{top}^2 + 2m_K^2 m_\pi^2 - m_\pi^4} \right\}, \quad (11)$$

where $m_{top}^2 = m_{\eta'}^2 + m_\eta^2 - 2m_K^2$. The result shows that the topological susceptibility vanishes if any of the quark masses $m_u = m_d$ or m_s is zero. This can be seen by using $m_\pi = 0$ for $m_u = m_d = 0$ and $m_\pi^2 = 2m_K^2$ for $m_s = 0$.

3. Instantons and the Witten-Veneziano relation: Large baryon density

The partition function described in the last section provides a simple and intuitive description of the η' prime mass and topological charge screening. The problem is that the theory is based on the assumption that the

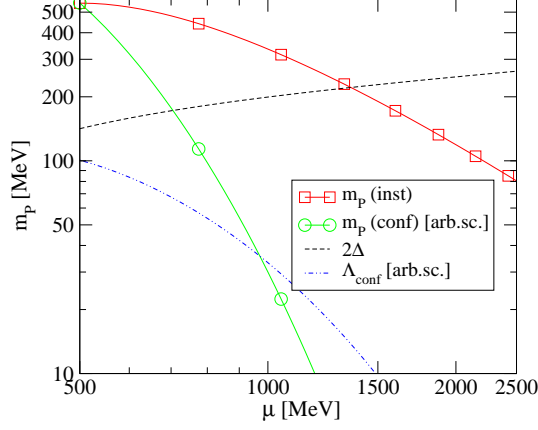


Figure 3. Flavor singlet pseudoscalar Goldstone boson mass in $N_c = 2$ QCD. The solid line marked by squares is the result of the instanton calculation. For comparison, the solid line marked by circles shows an estimate based on the assumption $m_P^2 f_P^2 \sim \sigma^2$, where σ is the string tension. The overall scale of this curve is not known. We also show the energy gap 2Δ and the confinement scale.

instanton liquid is dilute and weakly interacting. In this case the screening length $l \sim m_{\eta'}^{-1}$ is much larger than the typical distance between charges. In $N_c = 3$ QCD, however, the η' is heavy and the screening length is very short. Instead of $\rho \ll (N/V)^{1/4} \ll m_{\eta'}^{-1}$ we have $\rho \sim m_{\eta'}^{-1} < (N/V)^{1/4}$.

There is an interesting limit of QCD in which the dilute instanton liquid description can be rigorously justified. This is the case of QCD at large baryon density. It has long been known that large instantons are suppressed if the baryon density (or the temperature) is large. In the last few years it has also become clear that chiral symmetry remains broken at large baryon density²¹. This implies that there is a flavor singlet Goldstone boson, and that in the limit $m_q \rightarrow 0$ the mass of this mode is due to the anomaly.

In the following we shall discuss the case of two colors and flavors^{22,23,24}, but the results can be generalized to other values of N_c and N_f . At large baryon density the axial $U(1)$ symmetry is broken by a diquark condensate $\langle \epsilon^{\alpha\beta} \psi_L^\alpha \psi_L^\beta \rangle = -\langle \epsilon^{\alpha\beta} \psi_R^\alpha \psi_R^\beta \rangle$. Here, α, β are spinor indices. The condensate is a color and flavor singlet. The $U(1)_A$ Goldstone mode corresponds to fluctuations of the relative phase of the left and right handed condensates. The effective lagrangian for the singlet Goldstone boson is

$$L = f_P^2 [(\partial_0 \phi)^2 - v^2 (\partial_i \phi)^2] - V(\phi). \quad (12)$$

The decay constant and Goldstone boson velocity are not related to instantons and can be determined in perturbation theory. At leading order the result is²²

$$f_P^2 = \left(\frac{\mu^2}{8\pi^2} \right), \quad v^2 = \frac{1}{3}, \quad (13)$$

where μ is the baryon chemical potential. The potential $V(\phi)$ vanishes in perturbation theory but receives contributions from instantons, see Fig. 2. We find

$$V(\phi) = -A_P \cos(\phi + \theta), \quad (14)$$

where θ is the QCD theta angle. If the chemical potential is big, $\mu \gg \Lambda_{QCD}$, large instantons are suppressed and the coefficient A_P can be determined in perturbation theory. The result is^{22,23}

$$A_P = C_{2,2} 6\pi^4 \left[\frac{4\pi}{g} \Delta \left(\frac{\mu^2}{2\pi^2} \right) \right]^2 \left(\frac{8\pi^2}{g^2} \right)^4 \left(\frac{\Lambda}{\mu} \right)^8 \Lambda^{-2} \quad (15)$$

with

$$C_{N_c, N_f} = \frac{0.466 \exp(-1.679 N_c) 1.34^{N_f}}{(N_c - 1)!(N_c - 2)!}. \quad (16)$$

At large μ the superfluid gap Δ can also be determined in perturbation theory. The result is^{25,26,27,28}

$$\Delta = \frac{512\pi^4 \mu}{g^5} \exp \left(-\frac{2\pi^2}{g(\mu)} - \frac{\pi^2 + 4}{16} \right). \quad (17)$$

Using equ. (13-17) we can determine the mass of the pseudoscalar Goldstone boson. We have

$$m_P^2 = \frac{A_P}{2f_P^2}. \quad (18)$$

Note that we have not used the large N_c limit in order to derive this Witten-Veneziano relation. The result is exact in the limit $\mu \gg \Lambda_{QCD}$ even if $N_c = 2, 3$. Also note that A_P is the second derivative of the effective potential with respect to θ at $\theta + \phi = 0$ and is equal to the density of instantons. The vacuum energy, however, is determined by minimizing V with respect to ϕ and is independent of θ . This implies that χ_{top} is zero, as expected for QCD with massless fermions.

Equ. (18) predicts the density dependence of the flavor singlet Goldstone boson mass, see Fig. 3. This prediction can be tested using lattice simulations. We should note that formally, the prediction is only valid for $\mu \gg \Lambda_{QCD}$ but it is interesting to note that the result extrapolates to $m_P \sim 1$ GeV at zero baryon density.

4. The Instanton liquid at large N_c

In this section we describe a study of the instanton ensemble in QCD for different numbers of colors²⁹. We consider the partition function of a system of instantons in pure gauge theory

$$Z = \frac{1}{N_I! N_A!} \prod_I^{N_I+N_A} \int [d\Omega_I n(\rho_I)] \exp(-S_{int}). \quad (19)$$

Here, $\Omega_I = (z_I, \rho_I, U_I)$ are the collective coordinates of the instanton I and $n(\rho)$ is the semi-classical instanton distribution function³⁰

$$n(\rho) = C_{N_c} \left(\frac{8\pi^2}{g^2} \right)^{2N_c} \rho^{-5} \exp \left[-\frac{8\pi^2}{g(\rho)^2} \right], \quad (20)$$

$$C_{N_c} = \frac{0.466 \exp(-1.679 N_c)}{(N_c - 1)!(N_c - 2)!}, \quad (21)$$

$$\frac{8\pi^2}{g^2(\rho)} = -b \log(\rho\Lambda), \quad b = \frac{11}{3} N_c. \quad (22)$$

We have denoted the classical instanton interaction by S_{int} . If the instanton ensemble is sufficiently dilute we can approximate the instanton interaction as a sum of two-body terms, $S_{int} = \sum_{IJ} S_{IJ}$. For a well separated instanton-anti-instanton pair the interaction has the dipole structure¹⁰

$$S_{int} = -\frac{8\pi^2}{g^2} \frac{4\rho_I^2 \rho_A^2}{R_{IA}^4} |u|^2 (1 - 4 \cos^2 \theta). \quad (23)$$

Here $\rho_{I,A}$ are instanton radii and R_{IA} is the instanton-anti-instanton separation. The relative color orientation is characterized by a complex four-vector $u_\mu = \frac{1}{2i} \text{tr}(U_{IA} \tau_\mu^+)$, where $U_{IA} = U_I U_A^\dagger$ depends on the rigid gauge transformations that describe the color orientation of the individual instanton and anti-instanton and $\tau_\mu^+ = (\vec{\tau}, -i)$. We have also defined the relative color angle $\cos^2 \theta = |u \cdot \hat{R}|^2 / |u|^2$. The dipole interaction is valid if $R_{IA}^2 \gg \rho_I \rho_A$.

A strongly overlapping instanton-anti-instanton pair is not a semi-classical field configuration and we do not know how to treat it correctly. In practice we have chosen to deal with this problem by including a short range repulsive core

$$S_{core} = \frac{8\pi^2}{g^2} |u|^2 f \left(\frac{\rho_I^2 \rho_J^2}{R_{IJ}^4} \right). \quad (24)$$

The precise form of the function $f(x)$ is not very important. What is important for the N_c scaling is that the core is proportional to the classical action

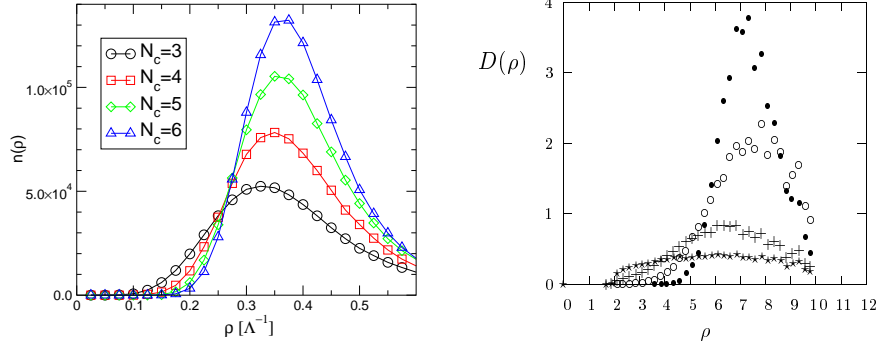


Figure 4. The figure on the left shows the instanton size distribution obtained from numerical simulations of the instanton ensemble in pure gauge QCD for different numbers of colors. The figure on the right shows lattice results reported by Teper at this meeting. The $\star + \circ \bullet$ symbols correspond to $N_c = 2, 3, 4, 5$.

$8\pi^2/g^2$ and that it includes the factor $|u|^2$ which ensures that instantons in commuting $SU(2)$ subgroups of $SU(N_c)$ do not interact.

The partition function equ. (19) is quite complicated and in general has to be analyzed using numerical methods. Before we describe numerical results we present a variational bound. Diakonov and Petrov¹² proposed to approximate the partition sum in terms of a variational single instanton distribution $\mu(\rho)$. For this ansatz the partition function reduces to

$$Z_1 = \frac{1}{N_I!N_A!} \prod_i^{N_I+N_A} \int d\Omega_I \mu(\rho_I) = \frac{1}{N_I!N_A!} (V\mu_0)^{N_I+N_A} \quad (25)$$

where $\mu_0 = \int d\rho \mu(\rho)$. The exact partition function is

$$Z = Z_1 \langle \exp(-(S - S_1)) \rangle, \quad (26)$$

where S is the full action, $S_1 = \log(\mu(\rho))$ is the variational estimate and the average $\langle \cdot \rangle$ is computed using the variational distribution function. The partition function satisfies the bound

$$Z \geq Z_1 \exp(-\langle S - S_1 \rangle), \quad (27)$$

which follows from convexity. The optimal distribution function $\mu(\rho)$ is determined from a variational principle, $(\delta \log Z)/(\delta \mu(\rho)) = 0$, where Z is computed from equ. (27). One can show that the variational result for the free energy $F = -\log(Z)/V$ provides an upper bound on the true free energy.

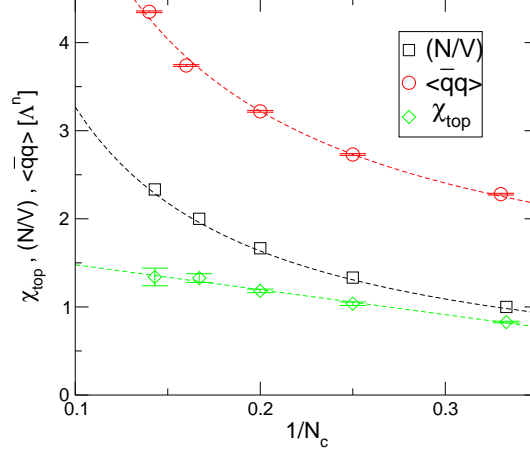


Figure 5. N_c dependence of the instanton density, the topological susceptibility and the quark condensate from numerical simulations of the instanton liquid in pure gauge QCD.

In order to compute the variational bound we have to determine the average interaction $\langle S_1 \rangle$. In the original work¹² the authors used the instanton interaction in the sum ansatz. The result is

$$\langle S_{int} \rangle = \frac{8\pi^2}{g^2} \gamma^2 \rho_I^2 \rho_J^2, \quad \gamma^2 = \frac{27}{4} \frac{N_c}{N_c^2 - 1} \pi^2. \quad (28)$$

For the “dipole plus core” interaction the result has the same dependence on N_c but the numerical coefficient is different. Note that the interaction contains a factor $N_c/(N_c^2 - 1) \sim 1/N_c$ which reflects the probability that two random instantons overlap in color space. Since the classical action scales as $S_0 \sim 1/g^2$ we find that the average interaction between any two instantons is $O(1)$. Applying the variational principle, one finds

$$\mu(\rho) = n(\rho) \exp \left[-\beta \gamma^2 \left(\frac{N \overline{\rho^2}}{V} \right) \rho^2 \right], \quad (29)$$

where $\beta = \beta(\overline{\rho})$ is the average instanton action and $\overline{\rho^2}$ is the average size. We observe that the single instanton distribution is cut off at large sizes by the average instanton repulsion. The instanton density and average size

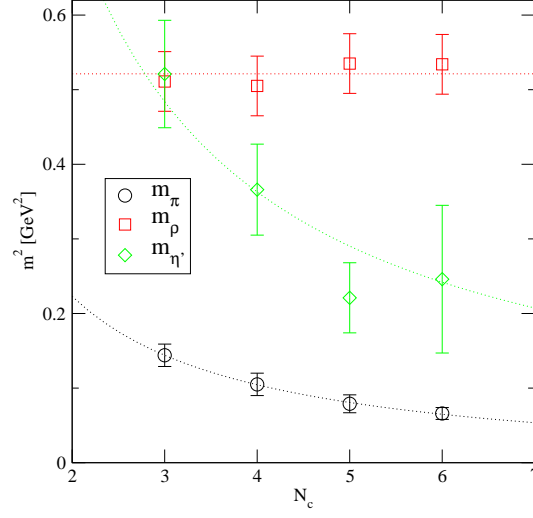


Figure 6. N_c dependence of the pion, the rho, and the eta prime mass from numerical simulations of the instanton liquid in pure gauge QCD. The dotted lines are fits to $m_\rho^2 \sim const$, $m_{\eta'}^2 \sim 1/N_c$ and $m_\pi^2 \sim c_0 + c_1/N_c$.

are given by

$$\frac{N}{V} = \Lambda^4 \left[C_{N_c} \beta^{2N_c} \Gamma(\nu) (\beta \nu \gamma^2)^{-\nu/2} \right]^{\frac{2}{2+\nu}}, \quad (30)$$

$$\overline{\rho^2} = \left(\frac{\nu V}{\beta \gamma^2 N} \right)^{1/2}, \quad \nu = \frac{b-4}{2}. \quad (31)$$

These results imply that

$$\left(\frac{N}{V} \right) \sim N_c, \quad \overline{\rho} \sim N_c^0. \quad (32)$$

Note that the instanton action is $O(N_c)$ in the large N_c limit, i.e. instantons remain semi-classical. The total density is not exponentially suppressed because an exponentially large factor associated with different instanton embeddings cancels the exponential suppression from the action. Also note that the density scales like the number of commuting subgroups of $SU(N_c)$ and the instanton packing fraction $\rho^4 N/(V N_c)$ is independent of N_c .

Numerical results are shown in Figs. 4-6. Fig. 4 shows the instanton size distribution. We observe that small instantons are suppressed as $N_c \rightarrow \infty$, but there is a critical size for which the number is independent of N_c and the total number scales as N_c . The results are consistent with the idea that

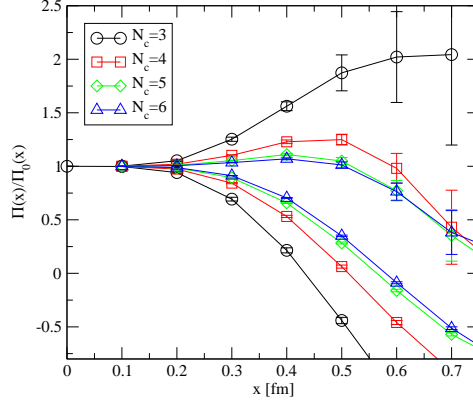


Figure 7. Correlation functions in the σ ($\bar{q}q$) and a_0 ($\bar{q}\tau^a q$) channel for different numbers of colors. The correlators are normalized to the free correlation functions and were calculated in a pure gauge instanton ensemble. The fact that the a_0 correlator becomes negative is an artefact of the quenched approximation which disappears as $N_c \rightarrow \infty$.

the size distribution slowly approaches a delta function^{31,32,33,34}. We also show lattice results reported by Teper at this meeting⁴. The lattice results are clearly consistent with our model calculation.

Fig. 5 shows the instanton density, the chiral condensate and the topological susceptibility. The instanton density and the chiral condensate scale as N_c . The topological susceptibility, on the other hand, goes to a constant as $N_c \rightarrow \infty$. This means that χ_{top} does not satisfy the relations $\chi_{top} \sim (N/V)$ expected for a dilute gas of instantons. In Fig. 6 we show the masses of the pion, the rho meson, and the eta prime meson. The results are consistent with the expectation $m_\rho^2 \sim N_c^0$ and $m_{\eta'}^2 \sim 1/N_c$.

5. Application: Scalar mesons and the large N_c limit

We saw in the previous section that the instanton liquid is consistent with the standard large N_c scaling relations. What is maybe even more important is that instantons can be used to understand corrections to the leading order large N_c results. It is well known, for example, that the OZI rule does not work equally well in all channels. The OZI rule is very well satisfied in the vector channel; the mixing is close to ideal, and the rho and omega meson are almost degenerate. In the scalar and pseudoscalar channels, on

the other hand, the OZI rule is badly violated. The eigenstates in the pseudoscalar sector are close to flavor, not mass, eigenstates and the mass difference between the pion and the eta prime meson is large. In the scalar sector we find a heavy iso-vector state, the a_0 , but a light iso-scalar, the σ -meson, which is strongly coupled to $\pi\pi$ states.

Jaffe suggested that the unusual properties of the light scalar mesons could be explained by assuming a large $(qq)(\bar{q}\bar{q})$ admixture^{35,36}. He observed that the spectrum of the flavor nonet obtained by coupling two anti-triplet scalar diquarks is inverted as compared to a standard $q\bar{q}$ nonet, and contains a light isospin singlet, a strange doublet, and a heavy triplet plus singlet with hidden strangeness. This compares very favorably to the observed light sigma, the strange kappa, and the heavier $a_0(980)$ and $f_0(980)$. It also explains why the a_0 and f_0 are strongly coupled to $K\bar{K}$ and $\pi\eta$.

Instantons are important because they can account for the observed pattern of OZI violating effects^{37,38}. There are no direct instanton effects in the vector channel, and as a result OZI violation is small. In the scalar and pseudoscalar channels, on the other hand, instantons lead to strong flavor mixing.

We have recently examined instanton contributions to scalar meson correlation functions in more detail³⁹. For some earlier work on the subject we refer the reader to⁴⁰. Fig. 7 shows the correlation functions in the sigma ($\bar{q}q$) and a_0 ($\bar{q}\tau^a q$) channel. For $N_c = 3$ we find a light ~ 600 MeV sigma state and a heavy ~ 1 GeV a_0 meson. When the number of colors is increased the light sigma state disappears and for $N_c = 6$ the sigma mass is also in the ~ 1 GeV range. We have also determined the off-diagonal sigma-pi-pi correlation function $\langle(\bar{q}q)(0)(\bar{q}\gamma_5\tau^a q)^2(x)\rangle$. We find that for $N_c = 3$ the sigma is strongly coupled to two-pion states, but for $N_c > 3$ the coupling becomes much smaller. These results are in agreement with a study based on chiral lagrangians⁴¹.

6. Supersymmetric gauge theories

In Sect. 4 we argued that the large N_c scaling behavior of the instanton contribution to QCD correlation functions agrees with the scaling of perturbative Feynman diagram. This result was based on fairly general arguments, but it did involve one important assumption regarding the effective instanton interaction that cannot be rigorously justified at this time. In order to study this problem in more detail it useful to consider supersymmetric generalizations of QCD.

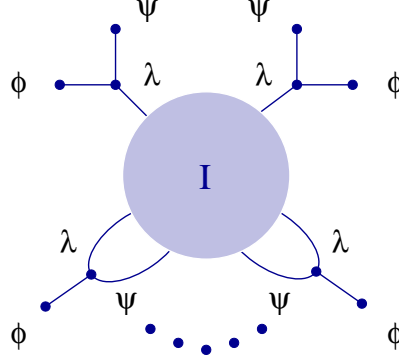


Figure 8. Instanton contribution to the superpotential in SUSY gluodynamics with $N_f = N_c - 1$ quark flavors. The dots denote $(2N_c - 4)$ pairs of quark ψ and gluino λ zero modes.

There is a significant literature on instanton effects in supersymmetric gauge theories at large N_c . Instantons in the AdS/CFT correspondence ($N = 4$ SUSY gauge theory) were studied in a series of papers by Bianchi et al and Dorey et al^{42,43,44}. The instanton contribution to the Seiberg-Witten superpotential in $N = 2$ SUSY gauge theory was originally studied by Finnell and Pouliot⁴⁵ and later generalized to arbitrary N_c by Klemm et al⁴⁶ and Douglas and Shenker⁴⁷.

In this contribution we would like to review some results that are relevant to SUSY gluodynamics ($N = 1$ SUSY gauge theory). This theory is interesting because it exhibits confinement and a fermion bilinear condensate. It was also recently argued that there is a new large N_c limit, called the orientifold large N_c expansion, which relates the quark condensate in $N_f = 1$ QCD to the gluino condensate in SUSY gluodynamics^{48,49}.

SUSY gluodynamics is defined by the lagrangian

$$L = -\frac{1}{4g^2} G_{\mu\nu}^a G_{\mu\nu}^a + \frac{i}{g^2} \lambda^{a\alpha} D_{\alpha\dot{\beta}} \lambda^{a\dot{\beta}}, \quad (33)$$

where λ is a Weyl fermion in the adjoint representation and $\alpha, \dot{\beta}$ are spinor indices. The theory has a $U(1)_A$ symmetry $\lambda \rightarrow e^{i\phi} \lambda$ which is broken by the anomaly. A discrete Z_{2N_c} subgroup is non-anomalous. The Z_{2N_c} symmetry is dynamically broken by the gluino condensate

$$\frac{1}{16\pi^2} \langle \text{Tr}[\lambda\lambda] \rangle = \Lambda^3 \exp\left(\frac{2\pi i k}{N_c}\right). \quad (34)$$

Here, $k = 0, 1, \dots, N_c - 1$ labels the N_c different vacua of theory and Λ is

the scale parameter defined by

$$\Lambda^3 = \mu^3 \frac{1}{g^2(\mu)} \exp\left(-\frac{8\pi^2}{g^2(\mu)N_c}\right). \quad (35)$$

The instanton solution in SUSY gluodynamics has $4N_c$ bosonic zero modes (4 translations, 1 scale transformation and $4N_c - 5$ rigid gauge rotations) and $2N_c$ fermion zero modes. Reviews of the SUSY instanton calculus can be found in^{50,51,52}. The collective coordinate measure is

$$\frac{2^{3N_c+2}\pi^{2N_c-2}\Lambda^{3N_c}}{(N_c-1)!(N_c-2)!} \int d^4z d\rho^2 (\rho^2)^{2N_c-4} d^2\eta d^2\zeta d^{N_c-2}\nu d^{N_c-2}\bar{\nu}, \quad (36)$$

where z denotes the instanton position and ρ is the instanton size. The Grassmann spinors η_α and ζ_α parameterize the so-called supersymmetric and superconformal zero modes, and the Grassmann numbers $\nu, \bar{\nu}$ parameterize the superpartners of the rigid gauge rotations.

There is no direct instanton contribution to the gluino condensate but the value of $\langle\lambda\lambda\rangle$ can be extracted from an indirect instanton calculation. The standard method consists of adding $N_f = N_c - 1$ quark flavors to the theory, and to consider the limit in which the squark fields have a large expectation value. In this case instantons are small and we can reliably compute their contribution to the superpotential. The result is^{53,54}

$$W_{eff}^{N_f=N_c-1, N_c} = \frac{(\Lambda_{N_c-1, N_c})^{b_0}}{\det_{N_f}(Q_f \tilde{Q}_{f'})} \quad (37)$$

where $Q_f, f = 1, \dots, N_f$ are quark superfields, $b_0 = 3N_c - N_f$ is the first coefficient of the beta function, and Λ_{N_f, N_c} is the scale parameter. Supersymmetry guarantees that equ. (37) is correct even if the squark vev is not large. The extra quark fields can be decoupled by sending the vev to infinity. The result is

$$W_{eff}^{0, N_c} = N_c \Lambda_{0, N_c}^3 \quad (38)$$

from which one can determine the gluino condensate equ. (34).

In SUSY gluodynamics there is a direct instanton contribution to the $(\lambda\lambda)^{N_c}$ correlation function. The result is independent of the relative coordinates and given by⁵⁴

$$\frac{1}{(16\pi^2)^{N_c}} \langle \text{Tr}[\lambda\lambda] \dots \text{Tr}[\lambda\lambda] \rangle = \frac{2^{N_c} \Lambda^{3N_c}}{(N_c-1)!(3N_c-1)}. \quad (39)$$

It is tempting to extract the gluino condensate from the N_c 'th root of equ. (39). This is sometimes called the strong-coupling instanton (SCI) calculation, in contrast to the weak-coupling instanton (WCI) result equ. (34).

The SCI result disagrees with the WCI by a factor that scales as N_c in the large N_c limit. It is not entirely clear why the SCI calculation fails, but the problem is likely related to the fact that the groundstate breaks a Z_{N_c} symmetry. This implies that the theory has to be defined carefully in order to pick out a unique ground state. In the WCI calculation the ground state is implicitly selected through the decoupling procedure.

As an alternative to the decoupling procedure one can define SUSY gluodynamics by compactifying the theory on $R^3 \times S^1$. Here, both fermions and bosons obey periodic boundary conditions and one can show that the gluino condensate is independent of the size of the compactified dimension. In the compactified theory the gauge symmetry is spontaneously broken by a non-zero expectation value of the gauge field A_4 . The corresponding gauge invariant order parameter is the Polyakov line (the holonomy) along the compact dimension. Instantons with a non-trivial holonomy were constructed by Kraan and van Baal⁵⁵. It turns out that these objects have constituent monopoles (dyons) with fractional topological charge

$$S_{dyon} = \frac{8\pi^2}{g^2 N_c}, \quad Q_{dyon} = \frac{1}{N_c}. \quad (40)$$

Each of these dyons has a pair of gluino zero modes that contribute directly to the gluino condensate^{56,57}

$$\frac{1}{16\pi^2} \langle \text{Tr}[\lambda\lambda] \rangle_{dyon} = \frac{1}{N_c} \Lambda^3. \quad (41)$$

Summing over the N_c constituent dyons one recovers the WCI result for the gluino condensate. This result supports the old idea that in the large N_c limit the relevant field configurations are not instantons but instantons constituents with fractional topological charge⁵⁸. These configurations are unsuppressed because they have action $S \sim O(1)$.

7. Conclusions

We have argued that it is possible for the instanton liquid model to have a smooth large N_c limit which is in agreement with scaling relations derived from Feynman diagrams. In this limit the density of instantons grows as N_c whereas the typical instanton size remains finite. Interactions between instantons are important and suppress fluctuations of the topological charge. As a result the $U(1)_A$ anomaly is effectively restored even though the number of instantons grows with N_c . Using variational arguments and numerical simulations we have shown that this scenario does not require

fine tuning. It arises naturally if the instanton ensemble is stabilized by a classical repulsive core. In this case we obtain a picture in which the instanton density is large but the instanton liquid remains dilute because instantons are not strongly overlapping in color space. Further investigations will have to show whether this scenario is indeed correct, but the lattice measurements of the instanton size distribution reported by Teper⁴ at this meeting are certainly encouraging.

We also emphasized that instantons provide a simple explanation of the observed pattern of OZI rule violating effects. Violations of the OZI rule are large in channels like the scalar and pseudoscalar mesons that receive direct instanton contributions. Instantons may also play a role in explaining regularities in hadron spectra that go beyond the naive quark model, such as diquark clustering in mesons and exotic baryons.

Finally we stressed that there are important lessons to be learned from generalizations of QCD. QCD at high baryon density provides a beautiful and rigorous realization of the instanton mechanism for generating the eta prime mass. Supersymmetric gluodynamics is an example for a theory in which instantons provide the essential non-perturbative input for the calculation of the fermion condensate. This calculation can now be linked, thanks to the orientifold large N_c limit, to the quark condensate in $N_f = 1$ QCD.

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